Electrostatic Probe Theory for Free-Molecular Spheres

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An analytical expression is given for the saturation current regimes of a current-voltage characteristic for a spherical free-molecular electrostatic probe surrounded by a finite sheath. The original derivation was based on the usual assumptions of a Maxwellian distribution of energies in a quiescent plasma. The current-voltage characteristic depends on parameters involving the sheath thickness and the electrostatic potential at the outer edge of the sheath. The latter parameter depends only on the ion/electron temperature ratio and is uniquely obtained from a transcendental equation. The sheath thicknesses are obtained from numerical solutions of the potential fields, and an empirical expression is derived to fit the numerical data. These results, which are for Debye lengths smaller than the probe radius, plus the "orbital limited" theory for larger Debye lengths cover most of the number density and temperature regimes obtained in laboratory and flowfield plasmas.

Nomenclature

e = magnitude of electron charge
I_j = total probe current of species j, amps
I_{oj} = $q_j N_{\infty j}(2\pi r_p^2)(k_j T/2\pi m_j)^{1/2}$ = thermal flux
k = Boltzmann constant
m_j = mass of particle of species j
N_j = number density of species j, cm^{-3}
$N_{\infty j}$ = ambient number density of species j, cm^{-3}
n_j = $N_j/N_{\infty j}$
q_j = charge of particle of species j
Q = $-q_a/q_r$
r = radial distance from probe center, cm
R = r/r_p
T_j = ambient temperature of species j, °K
Z = q_i/e
α = T_a/T_r
η = $q_r\phi/kT_r$
η_s = defined by Eq. (2)
$\bar{\eta}_s$ = $Q\eta_s/\alpha$
λ = charged particle mean free path
λ_D = $(kT_r/4\pi q_r^2 N_{\infty r})^{1/2}$ = Debye length, cm
ϕ = electrostatic potential, v
γ^2 = $[2/(1+\alpha/Q)](r_p/\lambda_D)^2 = 4.2 \times 10^{-2}[r_p^2 N_{\infty e}/(T_e + T_i/Z)]$
$\eta_{p\alpha}$ = $[(1+Q)/(1+\alpha)]\eta_p = 1.16 \times 10^4[(1+Z)/(T_e+T_i)]|\phi_p|$

Subscripts

a = attracted species
e = electrons
i = ions
p = probe surface
r = repulsed species
s = sheath
∞ = ambient value

I. Introduction

THE abundance of recent technical papers on electrostatic probes is indicative, not of substantially new analyses, but, rather, of refinements in previous work. This paper is also an attempt to present existing theoretical results in a more useful form. Here we combine the analyses of two previous papers to obtain simple analytical expressions for spherical electrostatic probe saturation currents. The basic derivation of the current-voltage characteristic was given by Kiel[1] in 1966, while some numerical solutions of the probe potential fields were described by Laframboise[2] in the same year. We have just recently obtained a more complete (and unpublished) set of the Laframboise results, thus enabling the following analysis to be completed.

Discussions of the experimental applications of theoretical results appear in many papers, e.g., Ref. 3. Consequently, the usual discussions covering plasma potential, floating potential, probe-support interaction, etc., will not be repeated here. The reader should keep in mind, however, the specific constraints on the following solution which include ambient Maxwellian distributions of both the ion and electron energies, probe potentials large compared to the floating potential ($\eta_p \gg 1$), free-molecular conditions ($\lambda > r_p$) for either species, and Debye lengths sufficiently small to ensure the formation of a sheath ($r_p/\lambda_D \gtrsim 3$). Throughout this report, the positive and negative charged particles will be identified as ions and electrons, respectively.

Only the results of the analysis will be reviewed here, since the details of the derivation are contained in Ref. 1. To obtain the spherical probe results we will follow a development which is analogous to that used previously in discussing the cylindrical probe.[4]

II. Spherical Probe Analysis

For conditions such that the Debye length is larger than the probe radius, the potential decays away from the probe surface approximately as r^{-2}, and a sheath region does not exist. For this large Debye length condition, an exact closed-form solution of the full kinetic equations of motion can be obtained. Such a solution is the original "orbital limited" result of Mott-Smith and Langmuir.[5] However, if the Debye length is smaller than the probe radius, the electrostatic potential of the probe is shielded from the surrounding plasma by a sheath region in which charge separation occurs. Within the sheath the electrostatic field tends to be very large and, thus, reduces the local potential from that of the probe to some value near the plasma potential over a very short distance. Beyond the outer edge of the sheath, charge separation does not occur, and the local potential asymptotically approaches the plasma potential. These two regions are conceptually depicted in Fig. 1.

Definition of the sheath boundary conditions is simply a prerogative of the analyst, since no true discontinuities occur in either the electric field or the charged particle density profiles. Typical definitions have been of two types: 1) the sheath edge is defined as that location at which the quasi-

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Table 1 Sheath potentials [solutions of Eq. (2) with $Q = 1$]

α	$\bar{\eta}_s$
0	2.17
0.01	2.03
0.1	1.45
0.5	0.68
1	0.44
2	0.26
5	0.12
10	0.064
∞	0

neutral solution becomes double valued,[6] or 2) that location at which the space-charge sheath obtains the plasma potential.[7] Both models extend beyond their region of validity, and, thus, both introduce unknown error. We define the sheath to exist within that region in which the electric field becomes so large that all attracted particles entering the sheath are collected by the probe. Mathematically this condition suggests that the ratio of the probe potential to the sheath potential must be very large. Taking the limit $(\eta_p/\eta_s) \to \infty$, the solution for the probe current obtained in Ref. 1 [see Eq. (38)] was given as

$$I_a = I_{oa}R_s^2(1 + \bar{\eta}_s) \quad (1)$$

where R_s defines the sheath radius and $(1 + \bar{\eta}_s)$ accounts for the inward acceleration of particles outside the sheath and where $\bar{\eta}_s = (Q/\alpha)\eta_s$. Assuming that the quasineutral region extends to the sheath edge, and equating attracted and repulsed particle densities at r_s, the following transcendental equation [see Eq. (41), Ref. 1] for the sheath potential is obtained:

$$2\exp(-\eta_s) = \mathrm{erfc}(\bar{\eta}_s^{1/2})\exp(\bar{\eta}_s) + 2(\bar{\eta}_s/\pi)^{1/2} \quad (2)$$

Selected values of $\bar{\eta}_s$ as a function of α are presented in Table 1 for $Q = 1$.

With $\bar{\eta}_s$ determined by Eq. (2), the one remaining unknown in Eq. (1) is R_s. An analytical determination of R_s has not been obtained. However, several numerical estimates of R_s obtained from Laframboise[2] were used in Ref. 1 to illustrate the validity of Eq. (1) for a limited number of cases. What has now been accomplished is the compilation of much of Laframboise's unpublished numerical results to obtain a large collection of sheath sizes as a function of several Debye length/probe radius ratios and probe potentials. Figure 2 illustrates the technique used to obtain R_s from his numerical data. All of his data for $T_i = T_e$ has been collapsed into the empirical correlation seen in Fig. 3. The Debye length and probe potential dependence have been incorporated into the values

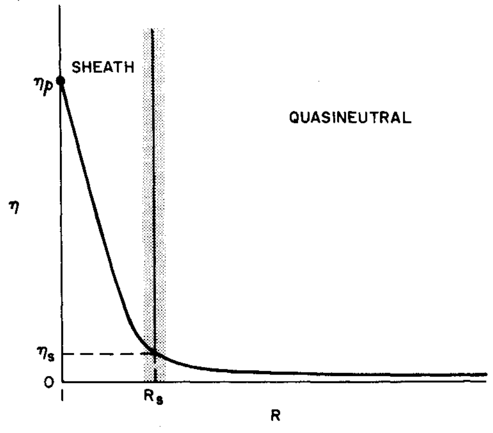

Fig. 1 Schematic of probe electrostatic field.

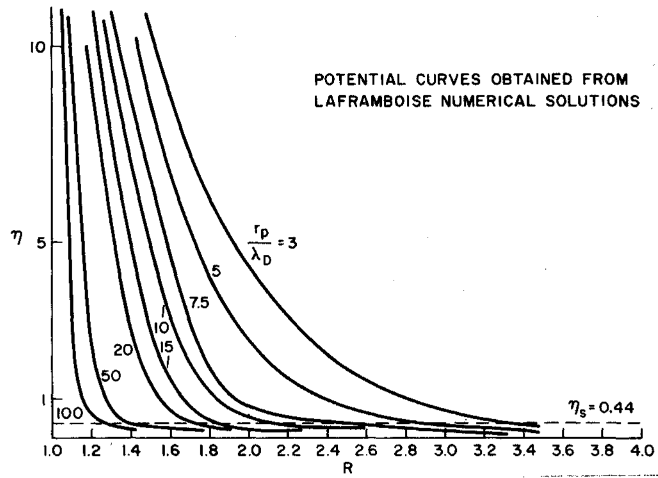

Fig. 2 Numerical solutions of electrostatic field for $\eta_p = 25$ and $T_i = T_e$.

of γ and $\eta_{p\alpha}$, respectively (see the Nomenclature for definitions). The equation for the straight line fit to the data in Fig. 3 is

$$(R_s)_{\alpha=1} = 1 + \gamma^{-2/3}\eta_{p\alpha}^{1/2} \quad (3)$$

The specific forms of γ and $\eta_{p\alpha}$ have been defined so that each is independent of the type of particle being collected, i.e., either ion or electron. (The definition of γ given here is slightly modified from that of Ref. 4.) In terms of the Laframboise definitions[2], γ and $\eta_{p\alpha}$ become

$$\gamma^2 = [2/(1 + T_+/T_-)](r_p/\lambda_{D-})^2$$

$$\eta_{p\alpha} = [2/(1 + T_+/T_-)](e\phi/kT_-)$$

Using Eq. (1) with Eqs. (2) and (3), the resulting probe currents for $T_i = T_e$ agree with those numerically calculated by Laframboise to within a few percent for all values of $r_p/\lambda_D \gtrless 3$, e.g., see Table 2 for $\alpha = 1$.

Although we have insufficient data on the electrostatic potential field solutions for cases other than $\alpha = 1$, our previous experience with the cylindrical probe[4] makes it very tempting to try to correlate existing numerical calculations for probe currents for $\alpha \neq 1$ with solutions using Eqs. (1), (2) and a value for R_s of the form $R_s = 1 + \tau(\alpha)\gamma^{-2/3}\eta_{p\alpha}^{1/2}$, where $\tau(\alpha)$ is some function of α only. Laboratory experience has shown that, in some "cold ion" cases, the ion current collection has been dominated by the electron temperature. Since $\bar{\eta}_s$ is finite even as $\alpha \to 0$, this experimental result suggests that the sheath area growth could be approximately related to $\alpha^{-1/2}$. Several attempts to empirically match the Laframboise currents for $\alpha \neq 1$ have obtained a value for $\tau(\alpha)$ which also satisfies this experimental experience. The result is that R_s can rather accurately be described by

$$R_s = 1 + [(1 + \alpha)/2\alpha]^{1/4}\gamma^{-2/3}\eta_{p\alpha}^{1/2} \quad (4)$$

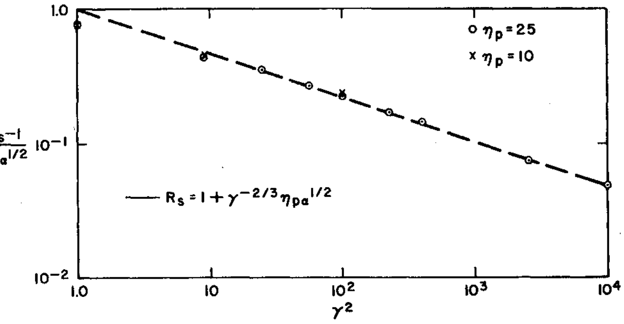

Fig. 3 R_s values from Eq. (2) and Laframboise solutions for $T_i = T_e$.

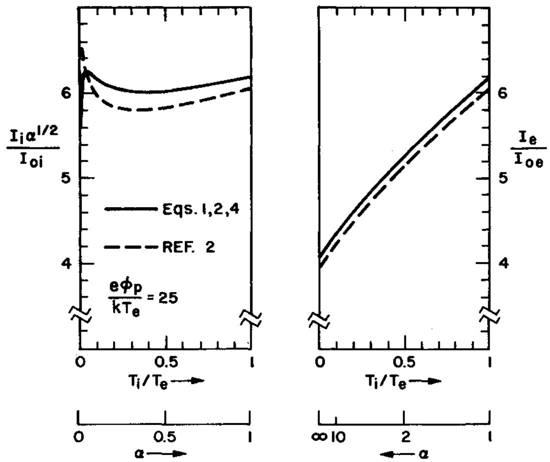

Fig. 4 Probe currents as a function of α.

for $0.01 < \alpha \leq \infty$. One comparison with the results of Ref. 2 is illustrated in Fig. 4, where the currents have been calculated by using Eq. (4) in Eq. (1) and $|e\phi_p/kT_e| = 25$. Note that Eq. (4) apparently breaks down for $\alpha < 0.01$.

III. Discussions and Conclusions

To within a few percent of the equivalent numerical calculations, Eq. (1) used with Eqs. (2) and (4) represents the saturation current regime of the current-voltage characteristic for spherical free-molecular probes operating under sheath conditions. This solution plus the "orbital limited" result,

$$I_a = I_{oa}(1 + \eta_p)$$

should be sufficient to obtain ion and electron densities to within experimental accuracy for most Debye lengths. The appropriate value of r_p/λ_D for switching from one model to another should be determined by whichever model gives the smaller current when all other parameters are held constant. Typically that value will be $r_p/\lambda_D \approx 1$–3.

For a spherical geometry, the expression for probe current derived here has no simple solution for the limiting case of $\alpha \rightarrow 0$ such as that given for the cylindrical probe in Ref. 4. (A detailed discussion of the $\alpha = 0$ case may be found in Ref. 2.) However, since Eq. (2) is valid for arbitrarily small values

Table 2 Typical current calculations using $|q_r\phi_p/kT_r| = 25$

α	γ	I_a/I_{oa}: Eqs. (1, 2, and 4)	I_a/I_{oa}: Ref. 2
1	3	16.7	13.6
1	5	10.6	9.7
1	7.5	7.6	7.3
1	10	6.2	6.1
1	15	4.8	4.7
1	20	4	4
1	50	2.7	2.7
1	100	2.2	2.2
0.05	13.8	28	28
0.10	13.5	20	19
0.50	11.5	8.5	8.3
1	10	6.2	6.1
2.0	11.5	5.3	5.2
10	13.5	4.4	4.3
∞	14.1	4	4

of α and since the empirical correction for α in Eq. (4) produces no large errors for $\alpha > 0.01$, there appears to be no real constraint in applying these solutions to usual laboratory or field plasmas.

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